# A WEAK SOLUTION CONCEPT WITH APPLICATIONS TO ELLIPTIC STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

**by**
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**Abstract**
We consider a weak solution concept of a differential equation $L(u) = 0$ by smoothing the individual terms by an approximative identity. We say that we have a solution of the equation if $L_n(u_n) \to 0$ in some sense, where $L_n$ and $u_n$ are smoothed versions of L and u. In this way it is possible to talk about solutions of equations involving products of distributions. In the first part we consider how this solution concept is related to classical solution concepts and discuss some simple examples. In the second part we use this theory to solve the SPDE div(a grad V)=$\dot{W}$ where $\dot{W}$=white noise, and a is strictly elliptic, bounded and measurable.

**Introduction**
It is well known that noise in stochastic partial differential equations (SPDEs) gives rise to very singular problems. Even for a linear problem the solutions are typically pure distributions (i.e. not functions), if the space-dimension is greater than or equal to two.

As an example consider the Poisson equation $\Delta u = \dot{W}$ where $\dot{W}$ is white noise in $\mathbb{R}^d$.

White noise is a random measure that is used to model singular forces. The typical force one has in mind is a force with extremely large values at some points, whereas space averages of the force will give moderate quantities. For a definition see [W] p. 269. It is possible to write down a distribution solution of the Poisson problem see [W] p. 417 and you get

$$u(\phi) = \int [\int K(x,y)\phi(x)dx]\dot{W}(dy)$$

where $K(x, y)$ is the Green function and the outer integral is defined as a White-noise integral. This is a well defined mathematical object. The point value

$$u(x) = \int K(x,y)\dot{W}(dy)$$

however, is meaningless.

It turns out that this behaviour is typical of SPDEs. This is a major obstacle in the study of non-linear problems, since you must expect to handle products of pure distributions. To bypass this difficulty, we have formulated a weak solution concept that (at least in principle) can be used to make sense of such objects. The idea is very simple. You replace all distributions by $C^\infty$ functions by convolving them with an approximative identity, and then let the approximative identity approach a $\delta$-function. The main result in the paper is the following

**Theorem**
Let $L = \text{div}(\text{a grad}(\cdot))$ on $\Omega$, a bounded domain in $\mathbb{R}^n$. Assume that L is strictly elliptic and that the coefficients $a = (a_{ij})$ are bounded measurable functions. If $G(x, y)$ is the Green function, then the distribution

$$V(\phi) = \int_\Omega \int_\Omega G(x,y)\phi(x)dx\dot{W}(dy)$$

solves the problem $LV = \dot{W}$ in the measurment sense with convergence in the sense of distributions (These concepts are explained in part one).

**Remark**
This problem, although linear, contains a singular product. It is necessary to make sense of the product of the pure distribution grad V with the non-smooth function a, and then differentiate on this product. The measurment method makes it possible to ignore the question of how to define the products. It takes care of div(a grad V)-$\dot{W}$ as a joint object, and if this object can be shown to be small, you say that you have a solution. We simply ignore the question of how to make sense of the individual terms.

The result above shows that this method can be applied to problems where classical methods fail to be effective. The interesting question of non-linear equations, however, we still know little about. The method is in principle applicable to such problems, but whether it is efficient or not remains to be seen.

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## Part 1 - The measurment method

Let $\Omega$ be a domain in $\mathbb{R}^d$, and let $\mathcal{D}(\Omega)$ denote the set of test functions on $\Omega$. We let $\{\delta_n\}_{n=1}^\infty$ denote an approximative identity. i.e. we choose $\delta_1 \in C_c^\infty(\mathbb{R}^d)$ s.t. $\delta_1 \geq 0, \int \delta_1 = 1$ and define $\delta_n(x) = n\delta_1(nx)$. If $u \in \mathcal{D}'(\Omega)$ is a distribution, it is well known that $u * \delta_n \to u$ in the sense of distributions. We call $u * \delta_n$ a measurment of u. In the following we will always assume that $\delta_1(0) \neq 0$ and that $\delta_1$ is rotation symmetric about the origin.

We want to make sense of very singular differential equations. As an example consider the "equation"

$$a_1 \frac{du}{dx} + a_2 u + a_3 = 0 \text{ where } a_1, a_2, a_3 \in \mathcal{D}'(\mathbb{R}).$$

As it stands, this is all nonsense. What we can do, however, is to introduce an "error-function" defined on every $u \in \mathcal{D}'(\mathbf{R})$. We let

$$E_n(u)(x) = a_1 * \delta_n(x) \cdot \frac{d}{dx}(u * \delta_n(x)) + a_2 * \delta_n(x) \cdot u * \delta_n(x) + a_3 * \delta_n(x)$$

This gives us $C^\infty$-functions on $\mathbf{R}$, and we let $\tau$ be a topology on such functions. We then make the following definition.

**Definition**
We say that $u \in \mathcal{D}'(\mathbf{R})$ solves the differential equation

$$a_1 \frac{du}{dx} + a_2 u + a_3 = 0$$

in the measurment sense with the $\tau$ topology iff $E_n(u)(x) \to 0$ in this topology.

Of course, this approach can be used to make sense of similar statements, linear or not, on any domain in $\mathbf{R}^d$. It is this formulation which we call the measurment method. We will now look at a few simple examples to see how to generalize some classical solution concepts.

**Example 1**
Assume that we have a smooth (function) solution of something like

$$\frac{du}{dx} + 2xu^2 = 0$$

We then define an errorfunction

$$E_n(u)(x) = \frac{d}{dx}(u * \delta_n(x)) + 2x * \delta_n(x)(u * \delta_n(x))^2$$

If u is a smooth function, we clearly have that $E_n(u)(x) \to \frac{du}{dx} + 2xu^2$ uniformly in x. It is then obvious to observe that a smooth ($C^1$) function solves the problem $\frac{du}{dx} + 2xu^2 = 0$ if and only if $E_n(u)(x) \to 0$ uniformly in x. Of course, there is nothing special about this particular equation, so the same statement is true for any problem, linear or not, as long as the solutions are smooth.

**Example 2**
Now consider a linear problem

$$Lu = a_1 \frac{\partial^2 u}{\partial x \partial t} + a_2 \frac{\partial u}{\partial x} + a_3 u = 0$$

with smooth ($C^\infty$) coefficients. We let

$$\begin{aligned} E_n(u)(x,t) &= a_1 * \delta_n(x,t) \frac{\partial^2}{\partial x \partial t}(u * \delta_n(x,t)) \\ &+ a_2 * \delta_n(x,t) \frac{\partial}{\partial x}(u * \delta_n(x,t)) \\ &+ a_3 * \delta_n(x,t)(u * \delta_n(x,t)) \end{aligned}$$

Now look at the first term

$$a_1 * \delta_n(x,t)\frac{\partial^2}{\partial x \partial t}(u * \delta_n(x,t)) = a_1 * \delta_n(x,t)(\frac{\partial^2 u}{\partial x \partial t} * \delta_n(x,t))$$

Since $a_1$ is smooth, we clearly have

$$a_1 * \delta_n \to a_1 \text{ in } C^\infty(\Omega) \text{ and } \frac{\partial^2 u}{\partial x \partial t} * \delta_n \to \frac{\partial^2 u}{\partial x \partial t} \text{ in } \mathcal{D}'(\Omega).$$

Then (see [R] p. 146)

$$a_1 * \delta_n(x,t) \cdot \frac{\partial^2 u}{\partial x \partial t} * \delta_n(x,t) \to a_1 \frac{\partial^2 u}{\partial x \partial t} \text{ in } \mathcal{D}'(\Omega)$$

The same thing happens with the remaining term so

$$E_n(u) \to a_1 \frac{\partial^2 u}{\partial x \partial t} + a_2 \frac{\partial u}{\partial x} + a_3 u \text{ in } \mathcal{D}'(\Omega)$$

Again this is a general principle, so we get the following

**Observation**

If L is linear with smooth ($C^\infty$) coefficients, the following statements are equivalent

(i) u solves $Lu = 0$ in the sense of distributions

(ii) $E_n(u) \to 0$ in the sense of distributions

**Example 3**

For non-linear problems the situation is not that obvious. These problems are not adapted to distribution theory. We want to show that the measurment method again gives a sensible generalization, so consider the following problem

$$u_t + (\frac{1}{2}u^2)_x = 0 \text{ on } \mathbb{R} \times \mathbb{R}^+$$

A classical solution concept for this equation is to consider functions $u(x,t)$ s.t.

$$\int\int u(x,t)\frac{\partial \phi}{\partial t} + \frac{1}{2}u(x,t)^2 \frac{\partial \phi}{\partial x} dx dt = 0$$

for all $\phi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^+)$. The problem has the natural error-function

$$E_n(u)(x,t) = \frac{\partial}{\partial t}(u * \delta_n(x,t)) + \frac{\partial}{\partial x}[\frac{1}{2}(u * \delta_n(x,t))^2]$$

Let us consider the statement $E_n(u) \to 0$ in the sense of distributions. Pick $\phi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^+)$. Then we have

$$\begin{aligned} E_n(u)(\phi) &= \int\int [\frac{\partial}{\partial t}(u * \delta_n(x,t)) + \frac{\partial}{\partial x}[\frac{1}{2}(u * \delta_n(x,t))^2]]\phi(x,t) dx dt \\ &= -\int\int u * \delta_n(x,t)\frac{\partial \phi}{\partial t} + \frac{1}{2}(u * \delta_n(x,t))^2 \frac{\partial \phi}{\partial x} dx dt \end{aligned}$$

If u is a reasonable function, we see that $E_n(u)(\phi) \to 0$ if and only if

$$\int\int u\frac{\partial\phi}{\partial t} + \frac{1}{2}u^2\frac{\partial\phi}{\partial x}dxdt = 0$$

Again we get that the solution concepts coincide.

**Solving meaningless equations**
The advantage of the measurment method is that it also gives meaningful statements for pure distributions. We will end this section with an example of an equation not solvable in any classical sense.

**Problem**
$\frac{dF}{dt} + \delta \cdot F = 0$ $\delta$=Dirac-delta at the origin. As an initial condition, we choose a $\psi_0 \in C_c^\infty(-\infty,-1)$ s.t. $\psi_0 \geq 0$ $\int \psi_0 = 1$ and require that $F * \psi_0(0) = 1$.

**Note: If F is a $C_c^\infty$-function, $\delta \cdot F$ does not mean $\delta(F)$, but rather the distribution $\psi \curvearrowright \delta(F\psi)$**

We put

$$E_n(F)(t) = \frac{dF}{dt} * \delta_n(t) + (\delta * \delta_n(t))(F * \delta_n(t))$$

We want to prove the following

**Theorem**

$$F(t) = \begin{cases} 1 & \text{if } t < 0 \\ \frac{1}{3} & \text{if } t > 0 \end{cases}$$

is the only distribution s.t. $E_n(F)(t) \to 0$ pointwise, uniformly on compacts outside the origin.

To get uniqueness we will have to assume some additional requirements on the measurment. By symmetry, we know that all derivatives $\delta_1^{(2k+1)}(0) = 0$. We will also assume that all derivatives $\delta_1^{(2k)}(0) \neq 0$ (It is easy to find such measurments. We leave it to the reader).

**Proof**
First of all, we see that if $t \neq 0$ and n is large enough, then

$$E_n(F)(t) = \frac{dF}{dt} * \delta_n(t).$$

If K is any compact set with $K \cap \{0\} = \emptyset$ and $\frac{dF}{dt} \neq 0$ on this set, then $\frac{dF}{dt} * \delta_n \to \frac{dF}{dt} \neq 0$ in the sense of distributions, and cannot possibly converge uniformly to zero. So for any compact set K as above we have

$$\frac{dF}{dt} = 0 \text{ on } K.$$

The only distributions satisfying this equation are the constant functions. The initial condition requires $F = 1$ to the left of the origin, so we consider

$$F = \begin{cases} 1 & \text{if } t < 0 \\ C & \text{if } t > 0 \end{cases} + \Delta \text{ supported at } \{0\}$$

Then we have

$$\begin{aligned} E_n(t)(F) = &\int_{-\infty}^{0} \frac{d\delta_n}{dt}(t-s)ds + C\int_{0}^{\infty} \frac{d\delta_n}{dt}(t-s)ds \\ &+ \frac{d\Delta}{dt} * \delta_n(t) \\ &+ \delta_n(t)[\int_{-\infty}^{0} \delta_n(t-s)ds + C\int_{0}^{\infty} \delta_n(t-s)ds] \\ &+ \delta_n(t)(\Delta * \delta_n(t)) \end{aligned}$$

We want $E_n(0)(F) \to 0$, and have

$$\begin{aligned} E_n(0)(F) = &\int_{-\infty}^{0} \frac{d\delta_n}{dt}(-s)ds + C\int_{0}^{\infty} \frac{d\delta_n}{dt}(-s)ds \\ &+ \delta_n(0)[\int_{-\infty}^{0} \delta_n(-s)ds + C\int_{0}^{\infty} \delta_n(-s)ds] \\ &+ \frac{d\Delta}{dt} * \delta_n(0) + \delta_n(0) \cdot \Delta * \delta_n(0) \\ = &-\delta_n(0) + C\delta_n(0) + \delta_n(0)(\frac{1}{2} + \frac{1}{2}C) \\ &+ \frac{d\Delta}{dt} * \delta_n(0) + \delta_n(0)(\Delta * \delta_n)(0) \\ = &\frac{1}{2}(3C-1)\delta_n(0) + \frac{d\Delta}{dt} * \delta_n(0) + \delta_n(0)(\Delta * \delta_n)(0) \end{aligned}$$

Since $\Delta$ is supported at the origin, we have [R] p. 150

$$\Delta = \sum_{k=0}^{N} a_k D^k \delta \quad D\Delta = \sum_{k=0}^{N} a_k (-1) D^{k+1} \delta$$

Then

$$\Delta * \delta_n(0) = \sum_{k=0}^{N} a_k (-1)^k D^k \delta_n(0)$$

$$D\Delta * \delta_n(0) = \sum_{k=0}^{N} a_k (-1)^{k+1} D^{k+1} \delta_n(0)$$

Since $\delta_n(x) = n\delta_1(nx)$ we also have

$$D^k\delta_n(0) = n^{k+1}\delta_1^{(k)}(0) \quad k = 0, 1, 2, \cdots$$

This gives

$$\begin{aligned} E_n(F)(0) &= \frac{1}{2}(3c-1) \cdot n\delta_1(0) \\ &+ \sum_{k=0}^{N} a_k(-1)^{k+1} n^{k+2}\delta_1^{(k+1)}(0) \\ &+ \sum_{k=0}^{N} a_k(-1)^k n^{k+2}\delta_1^{(k)}(0) \cdot \delta_1(0) \\ &= \frac{1}{3}(3c-1) \cdot n\delta_1(0) \\ &+ \sum_{k=0}^{N} a_k[\delta_1^{(k)}(0) \cdot \delta_1(0) - \delta_1^{(k+1)}(0)](-1)^k n^{k+2} \end{aligned}$$

If this converges to zero, then we must have

$$\frac{1}{3}(3c-1)\delta_1(0) = 0 \quad a_k(\delta_1^{(k)}(0)\delta_1(0) - \delta_1^{k+1}(0)) = 0 \quad k = 0, 1, 2, \cdots N$$

Since $\delta_1(0) \neq 0$ and for each k either $\delta_1^{(k)}(0)$ or $\delta_1^{(k+1)}(0) = 0$ this is not possible unless $c = \frac{1}{3}$ and all $a_k = 0$

$\square$

**Part II**

The problem

div[a grad V]="White noise".

Let L=div[a grad $(\cdot)$] on a bounded domain $\Omega$ in $\mathbb{R}^d$. We will assume that L is strictly elliptic on $\Omega$, and that the coefficients $a = [a_{ij}]$ are bounded measurable functions. See [G&T] p.167 for definitions. We assume zero boundary conditions and let $G(x, y)$ be the corresponding Green function. If $\psi \in L^2(\Omega)$, then [G&T], p.173 the problem

$$Lu = \psi$$

is weakly solvable with solution $u_\psi(x) = \int_\Omega G(x, y)\psi(y)dy$, and we have

$$\|u_\psi\|_{w^{1,2}(\Omega)} \leq C\|\psi\|_{L^2(\Omega)} i.e.$$

$u_\psi$ is in the Sobolev-space $W^{1,2}(\Omega)$ and

$$(*) \qquad \int_\Omega |u_\psi(x)|^2 + |Du_\psi(x)|^2 dx \leq C^2 \|\psi\|^2_{L^2(\Omega)}$$

We now define a distribution $V \in \mathcal{D}'(\Omega)$ by

$$V(\phi) = \int_\Omega \int_\Omega G(x,y)\phi(x) dx W(dy) \quad \phi \in \mathcal{D}(\Omega)$$

By the Ito-isometry and $(*)$ above, we have

$$E|V(\phi)|^2 = \int_\Omega |u_\phi(x)|^2 dx \leq C\|\phi\|^2_{L^2(\Omega)} \leq C \sup_{x\in\Omega} |\phi(x)|^2$$

(We allow the constant C to change value from step to step. Here we have assumed that $\Omega$ is bounded). This shows that the mapping $\phi \curvearrowright V(\phi)$ is well defined, almost linear and continuous in probability on $\mathcal{D}(\Omega)$. It then has a version in $\mathcal{D}'(\Omega)$. [W] p.332

We want to prove the following theorem.

**Theorem**
$V(\phi) = \int_\Omega [\int_\Omega G(x,y)\phi(x)dx]\dot{W}(dy)$ solves the problem

$$LV = \dot{W}$$

in the measurment sense with the topology of convergence in the sense of distributions. The error-function is the following

$$E_n(V)(x) = \text{ div } [a * \delta_n(x) \text{ grad } [V * \delta_n(x)]] - \dot{W} * \delta_n(x)$$

**Proof**
We pick $\phi \in \mathcal{D}(\Omega)$ and want to prove that $E_n(V)(\phi) \to 0$. Since $\dot{W}$ is a distribution, we know that

$$\dot{W} * \delta_n(\phi) \to \dot{W}(\phi) = \int_\Omega \phi(x)\dot{W}(dx).$$

We then consider the action of the term div $[a * \delta_n(x)$ grad $(V * \delta_n(x)]$ on $\phi$. We have

$$\text{div}[a * \delta_n(x) \text{ grad } (V * \delta_n(x))](\phi)$$
$$= \int_\Omega a * \delta_n(x) V * \text{ grad } \delta_n(x) \text{ grad } \phi(x) dx$$
$$= \int_\Omega a * \delta_n(x) \int_\Omega \int_\Omega G(\xi, n) \text{ grad } \delta_n(x-\xi) d\xi \dot{W}(d\eta) \text{ grad } \phi(x) dx$$

Fubini [J] p.4 or [W] p.297

$$= \int_\Omega [\int_\Omega \int_\Omega a * \delta_n(x) G(\xi,\eta) \text{ grad } \delta_n(x-\xi) \text{ grad } \phi(x) d\xi dx] \dot{W}(d\eta)$$

To prove the theorem, we have to prove that the term

$$\phi_n(\eta) = \int_\Omega \int_\Omega a * \delta_n(x) G(\xi, \eta) \text{ grad } \delta_n(x - \xi) \text{ grad } \phi(x) d\xi dx$$

is close to $\phi(\eta)$ in $L^2(\Omega)$, i.e. we want to prove that $\|\phi_n - \phi\|_{L^2(\Omega)} \to 0$. To prove this we pick $\psi \in L^2(\Omega)$ and estimate

$$\int_\Omega (\phi_n(\eta) - \phi(\eta))\psi(\eta) d\eta$$

It is then enough to prove that

$$\int_\Omega \phi_n(\eta)\psi(\eta) d\eta = \int_\Omega \phi(\eta)\psi(\eta) d\eta + O_n(\psi)$$

with

$$|O_n(\psi)| \leq O(n)\|\psi\|_{L^2(\Omega)} \text{ where } O(n) \to 0$$

independent of the choice of $\psi \in L^2(\Omega)$. For this we have

$$\begin{aligned}
&\int_\Omega \phi_n(\eta)\psi(\eta) d\eta \\
&= \int_\Omega \int_\Omega \int_\Omega a * \delta_n(x) G(x - \xi, \eta) \text{ grad } \delta_n(\xi) \text{ grad } \phi(x) d\xi dx \psi(\eta) d\eta \\
&\overset{\text{Fubini}}{=} \int_\Omega a * \delta_n(x) [\int_\Omega \int_\Omega G(x - \xi, \eta)\psi(\eta) d\eta \text{ grad } \delta_n(\xi) d\xi] \text{ grad } \phi(x) dx
\end{aligned}$$

We now want to estimate the integral in the middle. We put

$$\begin{aligned}
\tilde{u}_\psi(x) &= \int_\Omega \int_\Omega G(x - \xi, \eta)\psi(\eta) d\eta \text{ grad } \delta_n(\xi) d\xi \\
&= \int_\Omega u_\psi(x - \xi) \text{ grad } \delta_n(\xi) d\xi \\
&= \int_\Omega \text{ grad } u_\psi(x - \xi)\delta_n(\xi) d\xi
\end{aligned}$$

Now observe that for $f \in L^2(\Omega)$, we have

$$
\begin{aligned}
&\| \int_\Omega f(x-u)\delta_n(u)du\|^2_{L^2(\Omega)} \\
&= \int_\Omega \int_\Omega f(x-u)\delta_n(u)du \cdot \int_\Omega f(x-v)\delta_n(v)dudx \\
&= \int_\Omega \int_\Omega \int_\Omega f(x-u)f(x-v)dx\delta_n(u)\delta_n(v)dudv \\
&\le \int_\Omega \int_\Omega [\int_\Omega f^2(x-u)dx]^{\frac{1}{2}} [\int_\Omega f^2(x-v)dx]^{\frac{1}{2}} \delta_n(u)\delta_n(v)dudv \\
&\le \int_\Omega \int_\Omega \|f\|^2_{L^2(\Omega)}\delta_n(u)\delta_n(v)dudv = \|f\|^2_{L^2(\Omega)}.
\end{aligned}
$$

With $f =$ grad $u_\psi$ we get

$$\|\tilde{u}_\psi(x)\|_{L^2(\Omega)} \le \| \text{ grad } u_\psi\|_{L^2(\Omega)} \le C\|\psi\|_{L^2(\Omega)} \text{ by } (*).$$

We then have

$$
\begin{aligned}
\int_\Omega \phi_n(\eta)\psi(\eta)d\eta &= \int a(x)\tilde{u}_\psi(x) \text{ grad } \phi(x)dx \\
&\quad + \int (a * \delta_n(x) - a(x)) \text{ grad } \phi(x)\tilde{u}_\psi(x)dx
\end{aligned}
$$

Since $\|a * \delta_n - a\|_{L^2} \to 0$ [H] p.22

$$= \int_\Omega a(x)\tilde{u}_\psi(x) \text{ grad } \phi(x)dx + 0_n(\psi)$$

We also have

$$
\begin{aligned}
&\int_\Omega a(x)\tilde{u}_\psi(x) \text{ grad } \phi(x)dx \\
&= \int_\Omega a(x) \int_\Omega \text{ grad } u_\psi(x-\xi)\delta_n(\xi)d\xi \text{ grad } \phi(x)dx \\
&= \int_\Omega \int_\Omega a(x) \text{ grad } u_\psi(x-\xi) \text{ grad } \phi(x)dx\delta_n(\xi)d\xi \\
&= \int_\Omega \int_\Omega (a(x) - a(x-\xi)) \text{ grad } u_\psi(x-\xi) \text{ grad } \phi(x)dx\delta_n(\xi)d\xi \\
&+ \int_\Omega \int_\Omega a(x-\xi) \text{ grad } u_\psi(x-\xi) \text{ grad}\phi(x)dx\delta_n(\xi)d\xi \\
&= O_n(\psi) \\
&+ \int_\Omega \int_\Omega a(x-\xi) \text{ grad } u_\psi(x-\xi) \text{ grad } \phi(x-\xi)dx\delta_n(\xi)d\xi \\
&+ \int_\Omega \int_\Omega a(x-\xi) \text{ grad } u_\psi(x-\xi)( \text{ grad } \phi(x) - \text{ grad } \phi(x-\xi))dx\delta_n(\xi)d\xi \\
&= \int_\Omega \int_\Omega a(x-\xi) \text{ grad } u_\psi(x-\xi) \text{ grad}\phi(x-\xi)dx\delta_n(\xi)d\xi \\
&+ O_n(\psi) \\
&= \int_\Omega \int_\Omega a(x) \text{ grad } u_\psi(x) \text{ grad } \phi(x)dx\delta_n(\xi)d\xi \\
&+ O_n(\psi) \\
&= \int_\Omega \int_\Omega \psi(x)\phi(x)dx\delta_n(\xi)d\xi + O_n(\psi) \\
&= \int_\Omega \psi(x)\phi(x)dx + O_n(\psi)
\end{aligned}
$$

□

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